COLLATZ FUNCTION LIKE INTEGRAL VALUE TRANSFORMATIONS AND SUBSEQUENT MORPHOLOGICAL ANALYSIS

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Background of Collatz Conjecture

In modern mathematics, one of the interesting and most enigmatic unsolved mathematical problems is *Collatz Conjecture in number theory and discrete dynamical systems proposed by one celebrated mathematician* Lothar Collatz in 1937.

Although the problem on which the conjecture is built is remarkably simple to explain and understand, the nature of the conjecture and the behavior of this dynamical system for proving or disproving the conjecture have been altogether exceedingly difficult.

Collatz Conjecture

A function T is defined on $\mathbb{N}$ $\mathbb{N}$. T is defined as follows,

$$T(n) = 3n + 1; if\ n\ is\ odd$$

$$T(n) = \frac{n}{2}\ ; if\ n\ is\ even$$

The iterative scheme is introduced $X_{n+1} = T(X_n)$
There is a natural number such that $X_i = 1$ for all initial value X_0. This is what is known as *Collatz Conjecture*.

Collatz like function

- *We have defined a class of transformations named as Integral Value Transformations (IVT) where we have had a Collatz Like function (s) where we have been able to make proof for Collatz like conjecture in our context.*

Integral Value Transformations (IVT)

$$IVT^{p,k}{}_{i}$$

$where, p\ denote\ p\ adic\ number\ system$

$and\ k\ denote\ dimension\ of\ the\ space,$

$i\ denote\ the\ naming\ index.$

$For\ example,$

$Let\ f: \{0,1\}^n \rightarrow \{0,1\}$

$and\ f(0,0) = 0, f(O,1) = 0, f(1,0) = 0; f(1,1) = 1$

$IVT^{2,2}{}_{8}(a, b) = \{(f(a_1, b_1), f(a_2, b_2), \ldots, f(a_n, b_n)\}_2 = c$

$where, a = (a_1 a_2 a_3 \ldots a_n)_2$

$IVT^{2,2}{}_{8}$ $is\ named\ as$ $\boldsymbol{Carry\ Value\ Transformation}$ (CVT).

Formulation of IVTs

$$IVT^{2,3}{}_{\#}$$

$$f\colon \{0,1,2\}^n \to \{0,1,2\}$$

$Where$

$$f(0,0,0)=0;\ f(0,0,1)=1;\ f(0,1,0)=1;\ f(0,1,1)=0;$$

$$f(1,0,0)=2;\ f(1,0,1)=2;\ f(1,1,0)=0;\ f(1,1,1)=0$$

$$IVT^{2,3}{}_2(a,b,c)=\{(f(a_1,b_1,c_1),f(a_2,b_2,c_2),\dots,f(a_n,b_n,c_n)\}_2=d_{10}$$

$Where\ a=(a_1,a_2,a_3,\dots,a_n)$

Where # is the decimal value of the 00220220

Formulation of IVTs

$$IVT^{3,2}{}_{\#}$$

$$f:\{0,1,2\}^n \rightarrow \{0,1,2\}$$

$Where$

$$f(0,0)=0;\ f(0,1)=1;\ f(0,2)=1;\ f(1,0)=2;$$

$$f(1,1)=0;\ f(1,2)=0;\ f(2,0)=0;\ f(2,1)=2;\ f(2,2)=0$$

$$IVT^{2,3}{}_2(a,b)=\{(f(a_1,b_1),f(a_2,b_2),\dots,f(a_n,b_n)\}_2=d_{10}$$

$Where\ a=(a_1,a_2,a_3,\dots,a_n)$

Where # is the decimal value of the 020002110

IVT in 1 dimension…

Let us define the IVT in $\mathbb{N}_0$ in 2-adic number systems. There are 4 (2^{2^1}) one variable two state cellular automata rules. These are as follows:

Variable	f_0	f_1	f_2	f_3
0	0	1	0	1
1	0	0	1	1

$IVT^{2,1}{}_{\#}$ is mapped a non-negative integers to a non-negative integers.

$$IVT^{2,1}{}_0(a) = \big((f_0(a_n)f_0(a_{n-1}) \dots f_0(a_1)\big)_2 = b$$

$$IVT^{2,1}{}_1(a) = \big((f_1(a_n)f_1(a_{n-1}) \dots f_1(a_1)\big)_2 = b$$

$$IVT^{2,1}{}_2(a) = \big((f_2(a_n)f_2(a_{n-1}) \dots f_2(a_1)\big)_2 = b$$

and $IVT^{2,1}{}_3(a) = \big((f_3(a_n)f_3(a_{n-1}) \dots f_3(a_1)\big)_2 = b$

Where 'a' is a non-negative integers and $a = (a_n a_{n-1} \dots a_1)_2$ and 'b' is the decimal value corresponding to the binary number.

Algebraic relations of IVTs

$IVT^{2,1}{}_0(x) = 0$ for all nonnegative integers.

$IVT^{2,1}{}_1(x) = (2^s - 1) - x$ where x is an s bit number ion binary representation.

$IVT^{2,1}{}_2(x) = x$; for all non-negative integer x.

$IVT^{2,1}{}_3(x) = 2^s - 1$; x is an s bit number in binary representation.

Now, $IVT^{2,1}{}_1(x) = (2^s - 1) - x$

$IVT^{2,1}{}_1(x) = (2^s - 1) - IVT^{2,1}{}_2(x)$

$IVT^{2,1}{}_2(x) = IVT^{2,1}{}_3(x) - IVT^{2,1}{}_1(x)$

i.e. $IVT^{2,1}{}_3(x) = IVT^{2,1}{}_1(x) + IVT^{2,1}{}_2(x)$ for all non-negative integers x.

Therefore, the relation becomes $\mathbf{IVT^{2,1}{}_3 = IVT^{2,1}{}_1 + IVT^{2,1}{}_2}$

Collatz like Conjecture

Let us consider an iterative scheme as $X_{n+1} = f(X_n)$ where f is a function from $\mathbb{N}_0$ to $\mathbb{N}_0$.

If we consider f as three IVTs' then we have X_{n+1}=0, X_0, $2^s - 1$ for all X_0 (s bit in binary representation) corresponding to $IVT^{2,1}{}_0$, $IVT^{2,1}{}_2$ and $IVT^{2,1}{}_3$ respectively. So, iterative scheme for these three transformations are basically static systems. But the transformation $IVT^{2,1}{}_1$ shows a significant dynamism in the iterative sequence. Let us take one example, let X_0 be 19. 19 (=10011_2) is a five (s) bit number. Therefore 19 maps to $(2^5 - 1) - 19 = 12$.

Iterative convergence

X_0	*Iterative sequences*
0	0
1	0
2	1, 0
3	0
4	3, 0
5	2, 1, 0
6	1, 0
7	0
8	7, 0
9	6, 1, 0
10	5, 2, 1, 0
11	4, 3, 0
12	3, 0
13	2, 1, 0
14	1, 0
15	0
16	15, 0
17	14, 1, 0
18	13, 2, 1, 0
19	12, 3, 0
20	11, 4, 3, 0

Lemma

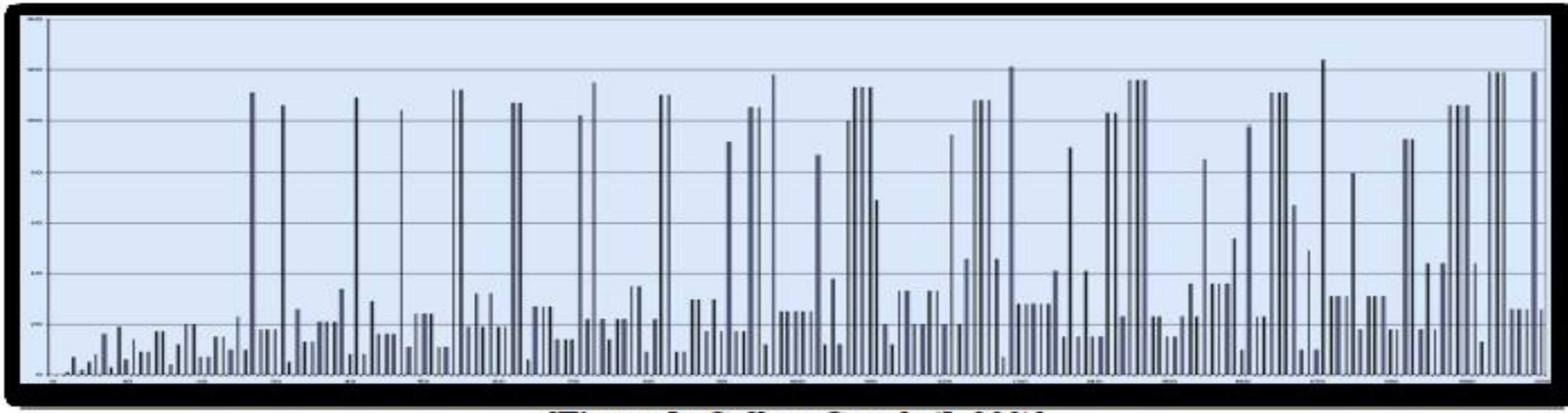
(I) *For any non-negative integer of the form* $X_0 = 2^n + P$, $IVT^{2,1}{}_1(X_0) = 2^n - (P+1)$ *for some non-negative integer.*

(II) *For any non-negative integer of Merseene form* $X_0 = 2^n - 1$, $IVT^{2,1}{}_1(X_0) = 0$.

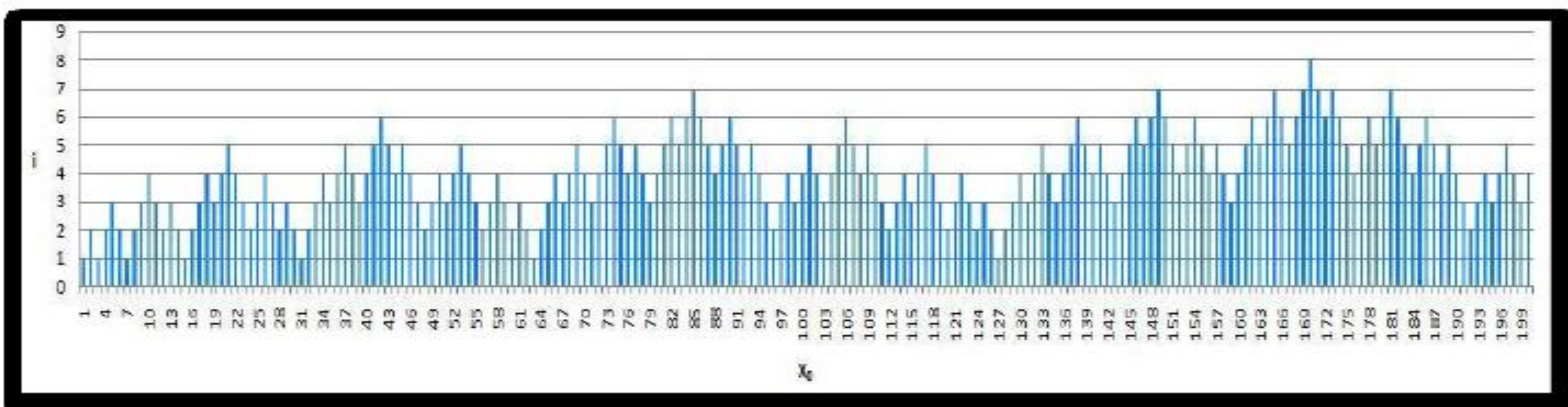
Main Result

The iterative scheme $\{X_n\}$ converges to 0 for any given X_0 where $X_{n+1} = IVT^{2,1}{}_1(X_n)$

Convergence behavior of Collatz and IVT

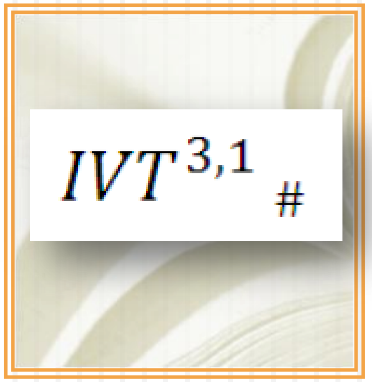


[Figure 1: Collatz Graph (1-200)]



[Figure 2: $IVT^{2,1}{}_1$ Graph (1-200)]

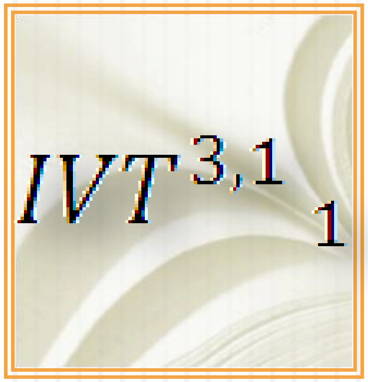
$IVT^{3,1}{}_{\#}$



Clearly there are $3^{3^1} = 27$ integral value transformations.

In this case $\{IVT^{3,1}{}_1, IVT^{3,1}{}_3, IVT^{3,1}{}_9\}$ is basis for the $IVT^{3,1}{}_{\#}$ system.

Algebraic form of IVTs



$\mathbf{IVT^{3,1}{}_{1}}$

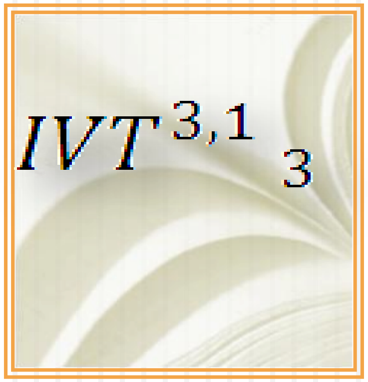
$$\mathrm{IVT}^{3,1}{}_{1}(\mathrm{n}) = \frac{3^p - 1}{2} - \left(\sum_{i=0}^{p-1} 3^i\right) : n = \sum_{i=0}^{p} 3^i$$

$$= \frac{3^p - 1}{2} - \left(\sum_{i=0}^{p-1} 3^i\right) : n = 2.\left(\sum_{i=0}^{p} 3^i\right)$$

$$= \frac{3^h - 1}{2} - \left[\left(\sum_{i=0}^{h-1} 3^i\right) + \left(\sum_{j=0}^{k} 3^i\right)\right] : n = \left(\sum_{i=0}^{h} 3^i\right) + 2.\left(\sum_{i=0}^{k} 3^i\right), h > k$$

$$= \frac{3^p - 1}{2} - \left[\left(\sum_{i=0}^{h} 3^i\right) + \left(\sum_{j=0}^{p-1} 3^i\right)\right] : n = \left(\sum_{i=0}^{h} 3^i\right) + 2.\left(\sum_{i=0}^{p} 3^i\right), h < k$$

Contd…



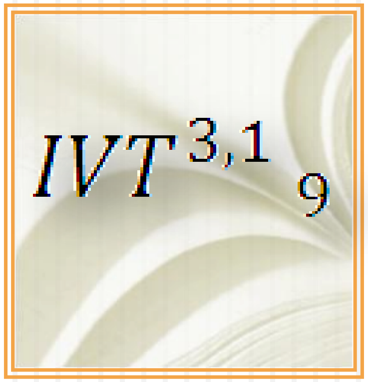
$\mathbf{IVT^{3,1}{}_{3}}$

$$\mathbf{IVT^{3,1}{}_{3}}(n) = \left(\sum_{j=0}^{p} 3^{i}\right) \; when \; n = \left(\sum_{j=0}^{p} 3^{i}\right)$$

$$= 0 \; when \; n = 2.\left(\sum_{j=0}^{p} 3^{i}\right)$$

$$= \left(\sum_{j=0}^{p} 3^{i}\right) \; when \; n = \left(\sum_{j=0}^{p} 3^{i}\right) + 2.\left(\sum_{k=0}^{q} 3^{i}\right)$$

Contd…



$\mathbf{IVT^{3,1}{}_{9}}$

$$\mathrm{IVT}^{3,1}{}_{9}(\mathrm{n}) = 0 \; when \; n = \left(\sum_{j=0}^{p} 3^{i}\right)$$

$$= \left(\sum_{j=0}^{p} 3^{i}\right) when \; n = 2.\left(\sum_{j=0}^{p} 3^{i}\right)$$

$$= \left(\sum_{j=0}^{p} 3^{i}\right) when \; n = \left(\sum_{j=0}^{p} 3^{i}\right) + 2.\left(\sum_{k=0}^{q} 3^{i}\right)$$

Contd…

The algebraic formulations of the remaining IVTs can be defined as a linear combination of the above three IVTs. Each $IVT^{3,1}{}_{\#}$ satisfies the following linear combination.

$$IVT^{3,1}{}_{\#} = a.\ IVT^{3,1}{}_{1} + b.\ IVT^{3,1}{}_{3} + c.\ IVT^{3,1}{}_{9}$$

$$\text{where } a, b, c \in \{0,1,2\}$$

satisfies the following relation:

$$\# = a + 3b + 9c$$

Collatz behavioral IVTs

$IVT^{3,1}{}_{0}$

$IVT^{3,1}{}_{1}$

$IVT^{3,1}{}_{2}$

$IVT^{3,1}{}_{6}$

$IVT^{3,1}{}_{7}$

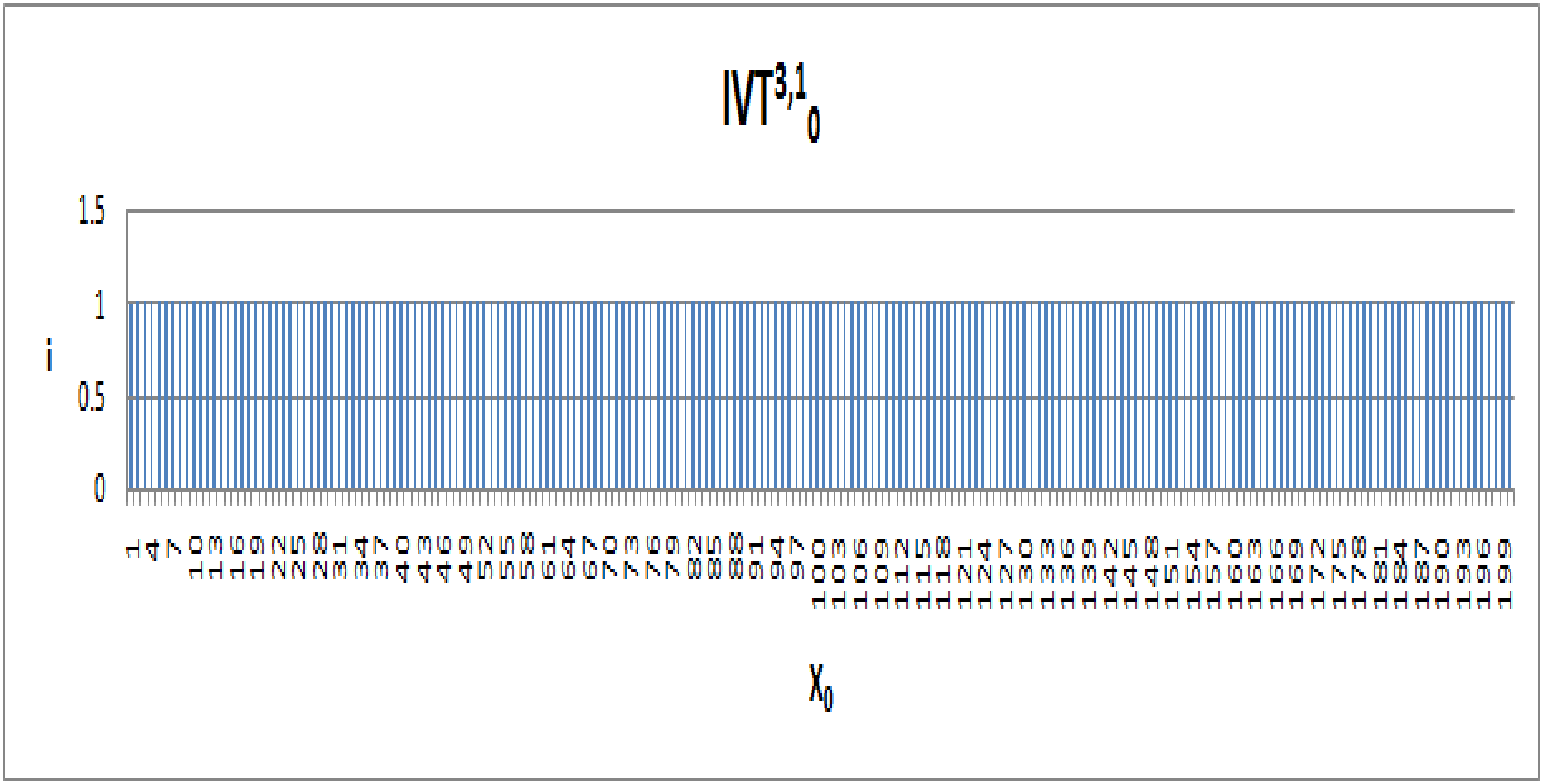
$IVT^{3,1}{}_{8}$

$IVT^{3,1}{}_{9}$

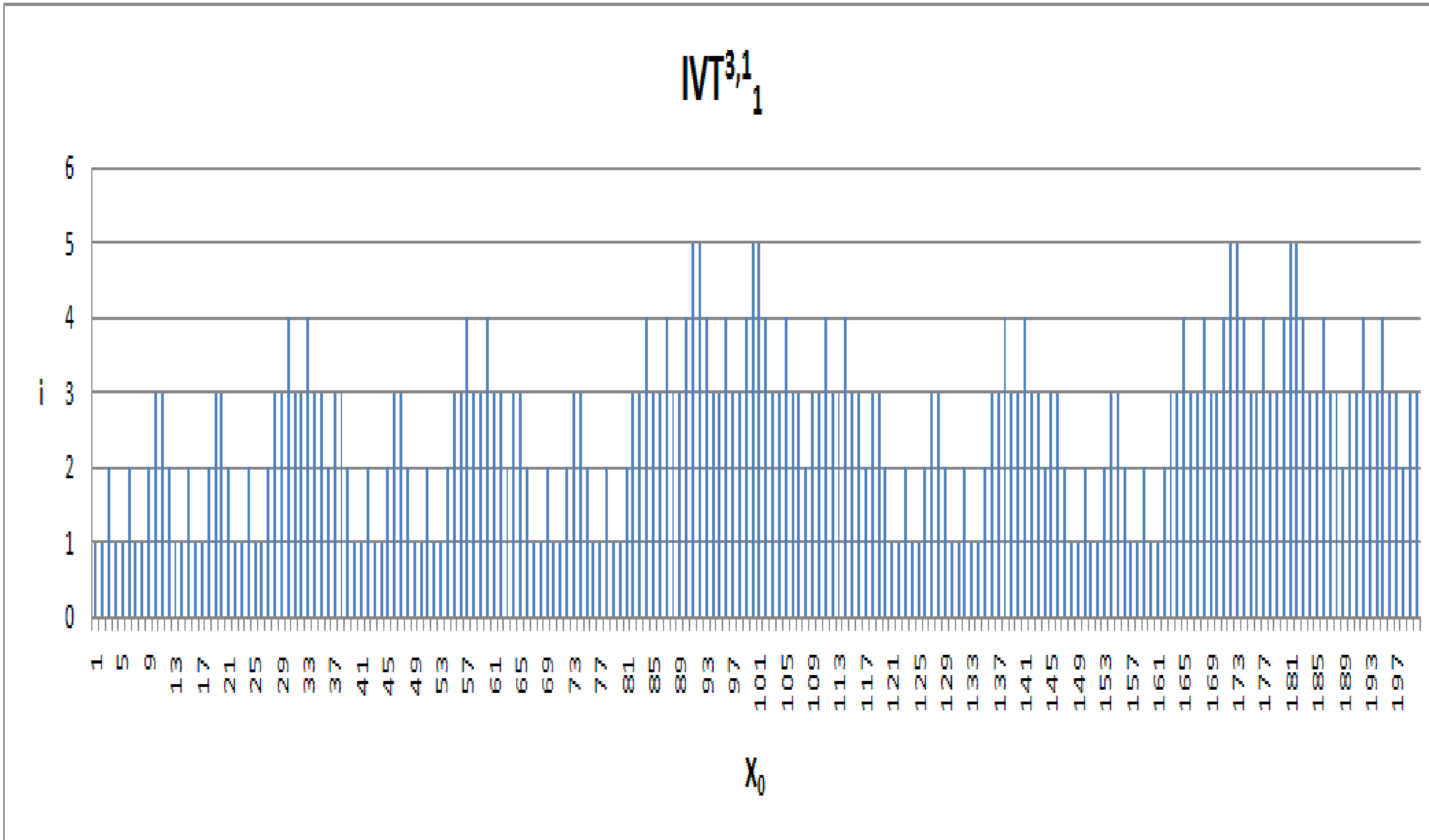
$IVT^{3,1}{}_{10}$

$IVT^{3,1}{}_{11}$

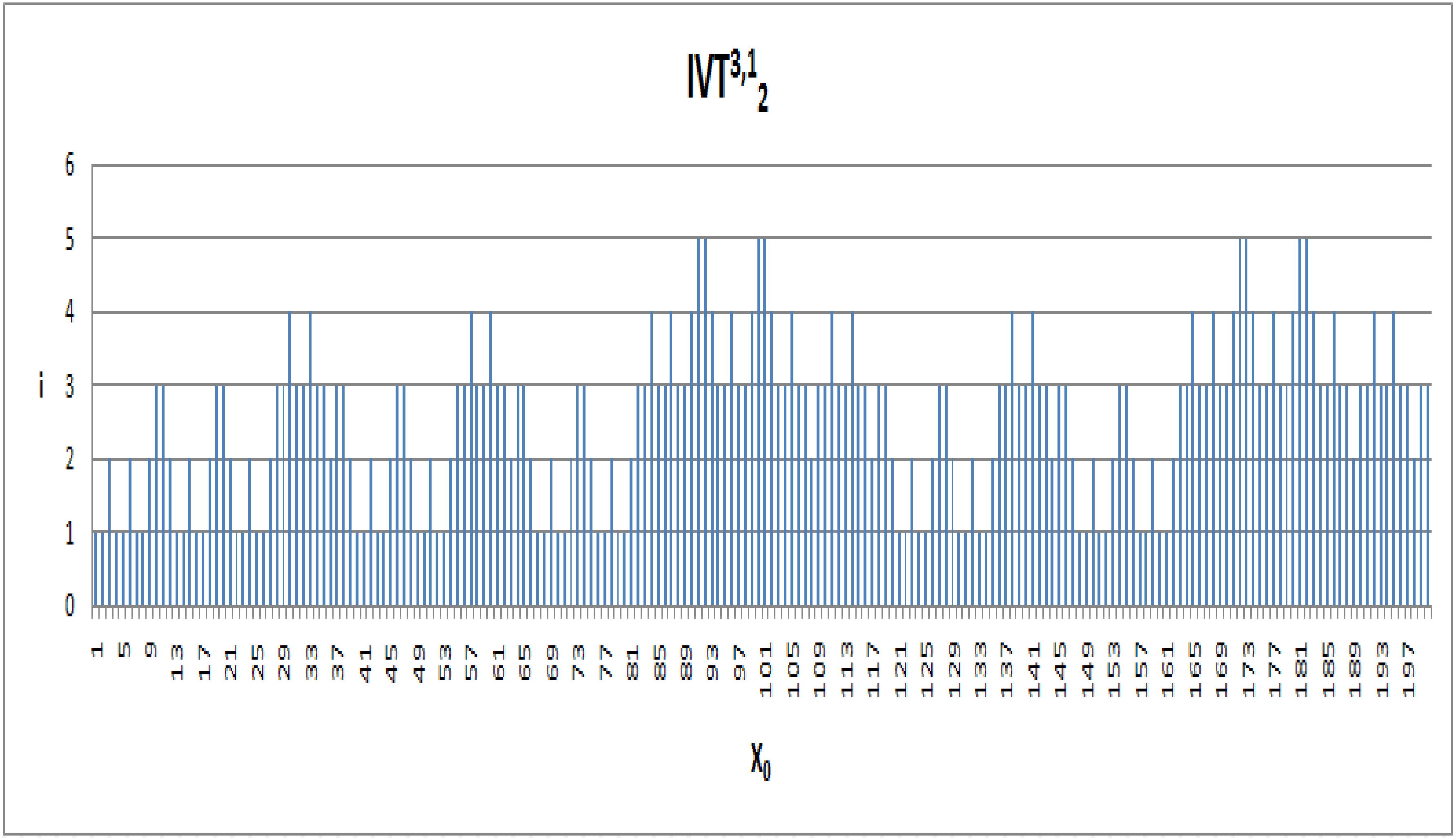
Convergence Graph of $IVT^{3,1}{}_0$



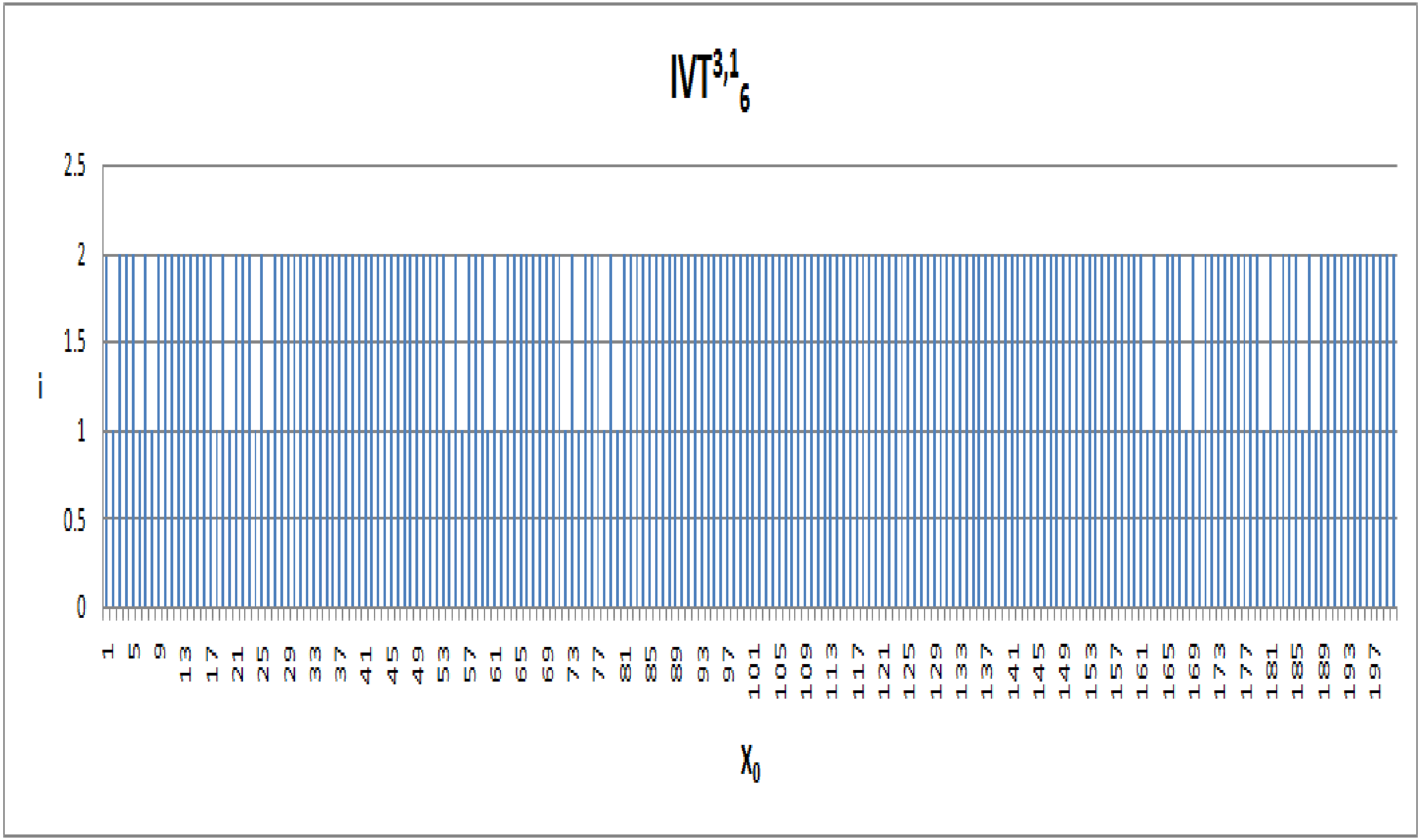
Convergence Graph of $IVT^{3,1}{}_1$



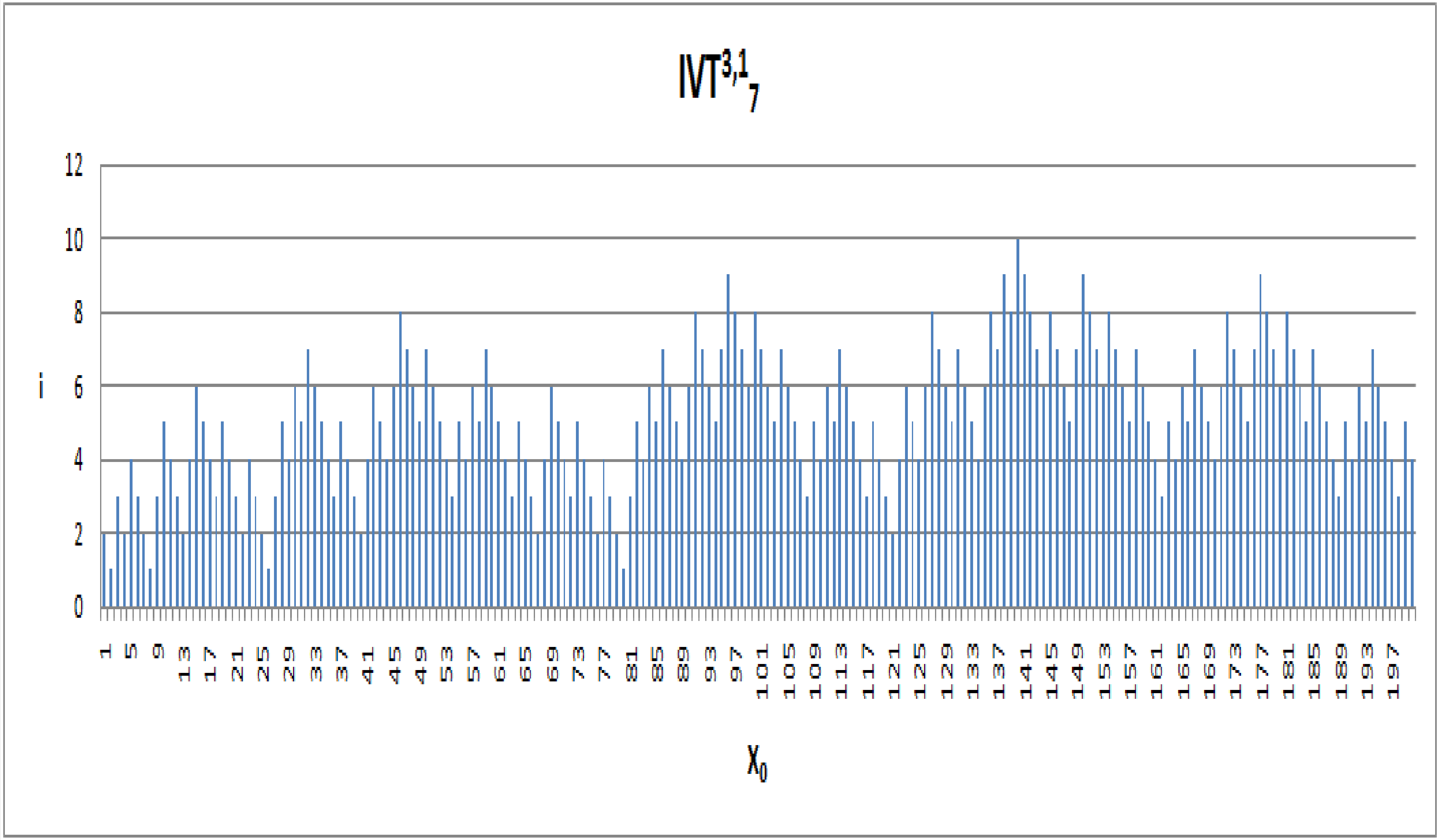
Convergence Graph of $IVT^{3,1}{}_{2}$



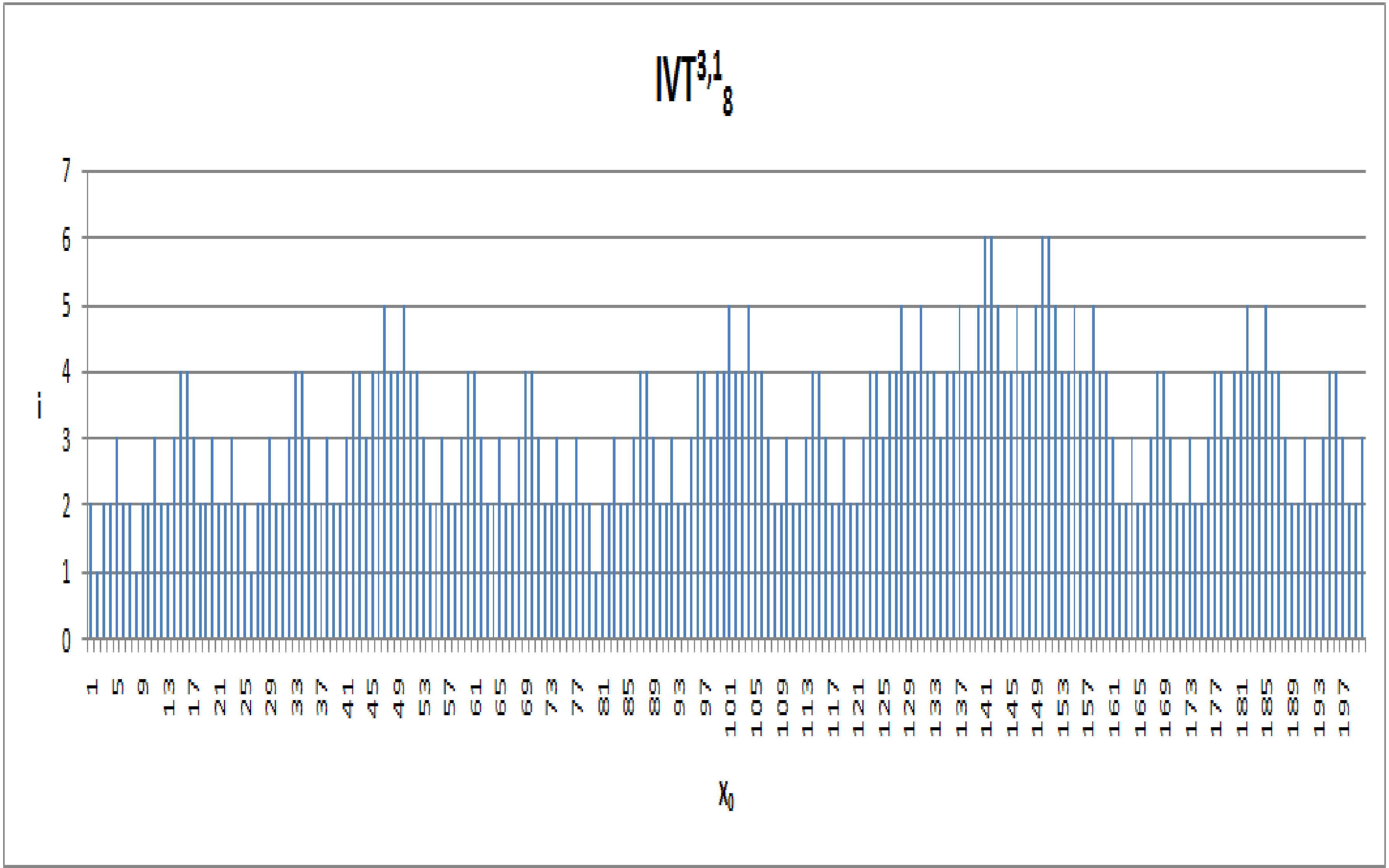
Convergence Graph of $IVT^{3,1}{}_{6}$



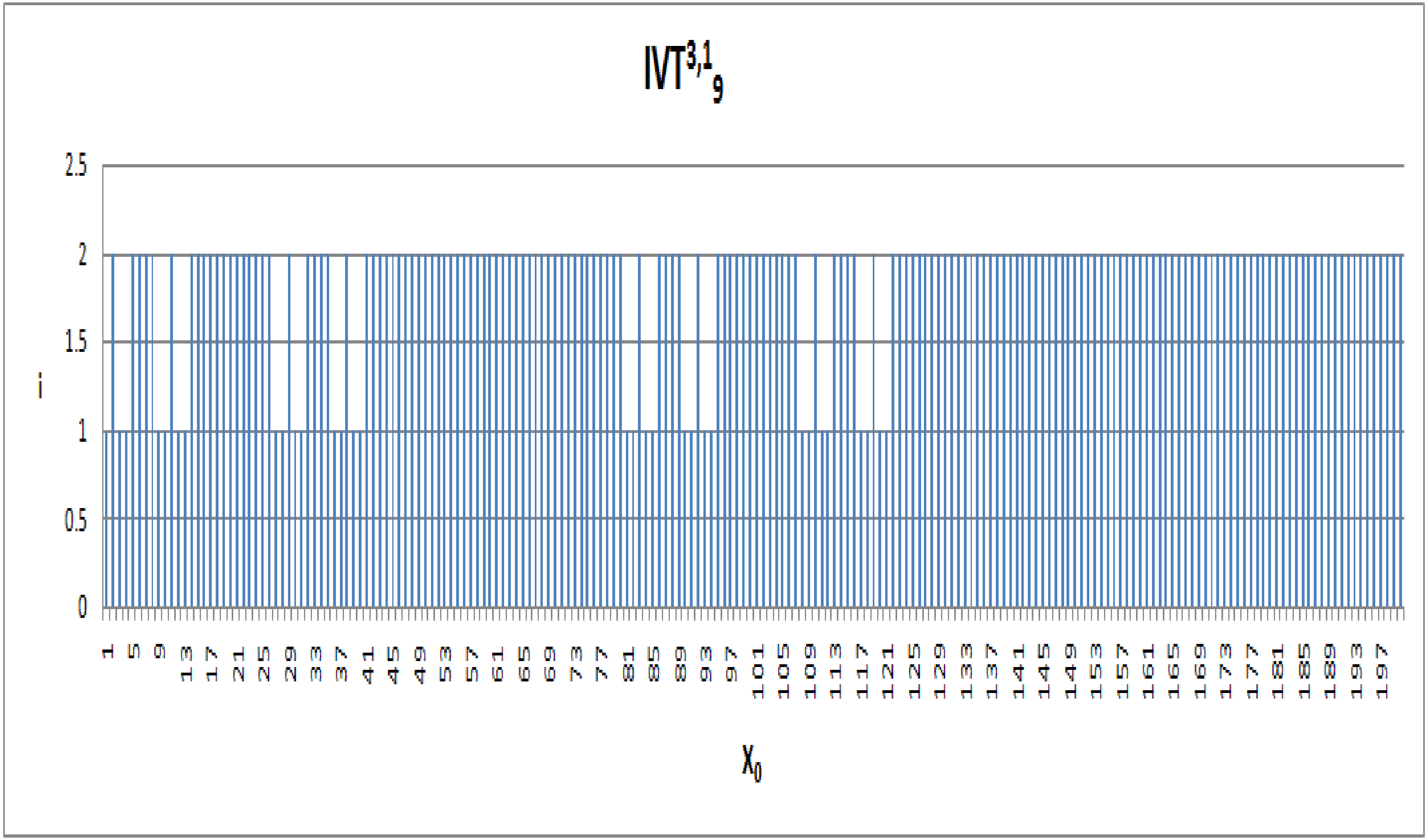
Convergence Graph of $IVT^{3,1}{}_7$



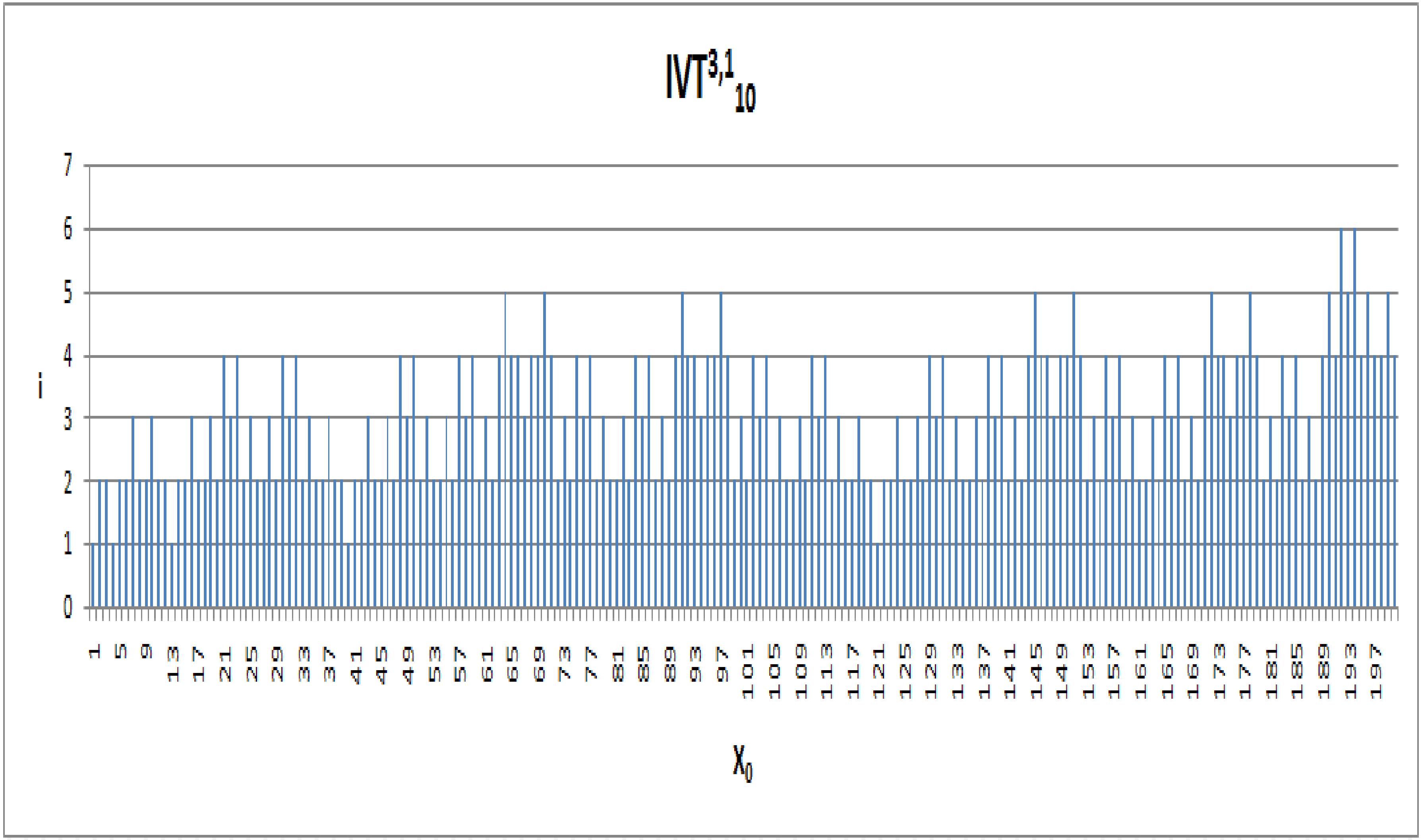
Convergence Graph of $IVT^{3,1}{}_8$



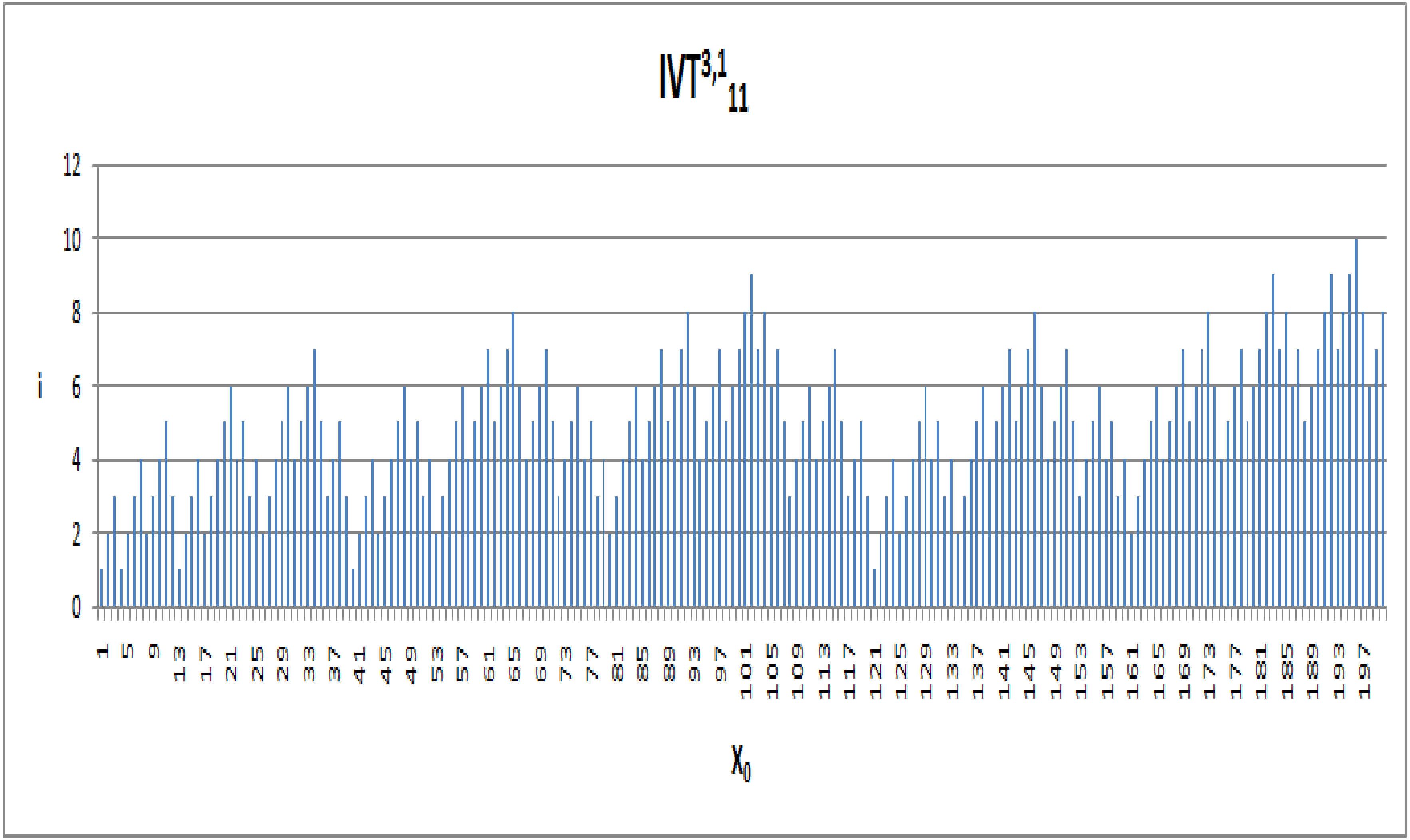
Convergence Graph of $IVT^{3,1}{}_9$



Convergence Graph of $IVT^{3,1}{}_{10}$



Convergence Graph of $IVT^{3,1}{}_{11}$



Morphology can enlighten the convergence behavior of each IVT

What we have observed :

In case of Collatz function, the convergence graph shows very slow rate of convergence for different X0's whereas in our IVT domain, there are Collatz like transformations for which the convergence graph shows very rapid convergence.

In our cases, not more than 10 iterations required to converge to 0 for set {0,1 ,2 …200} whereas the number 27 takes 111 iteration for converging to 1 in the Collatz function.

- This kind of convergence behavior we would like to translate in terms of Mathematical morphology.

Thanks